Exercise 1B

1 $a = 2 + \frac{1}{2}x$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = 2 + \frac{1}{2}x$$

$$\frac{1}{2}v^2 = \int\left(2 + \frac{1}{2}x\right)\mathrm{d}x = 2x + \frac{x^2}{4} + A$$

At $x = 0,\ v = 5$

$$\frac{1}{2} \times 25 = 0 + 0 + A \Rightarrow A = \frac{25}{2}$$

$$\frac{1}{2}v^2 = 2x + \frac{x^2}{4} + \frac{25}{2}$$

$$v^2 = \frac{x^2}{2} + 4x + 25$$

2 $a = -4x$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = -4x$$

$$\frac{1}{2}v^2 = \int(-4x)\,\mathrm{d}x = -2x^2 + A$$

At $x = 2,\ v = 8$

$$\frac{1}{2} \times 64 = -8 + A \Rightarrow A = 40$$

$$\frac{1}{2}v^2 = -2x^2 + 40$$

$$v^2 = 80 - 4x^2$$

$$v = \pm\sqrt{(80 - 4x^2)}$$

3 $a = \frac{4}{x^2}$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = \frac{4}{x^2}$$

$$\frac{1}{2}v^2 = \int(4x^{-2})\,\mathrm{d}x = -4x^{-1} + A = A - \frac{4}{x}$$

At $x = 2,\ v = 6$

$$\frac{1}{2} \times 36 = A - 2 \Rightarrow A = 20$$

$$\frac{1}{2}v^2 = 20 - \frac{4}{x}$$

When $v = 0$

$$0 = 20 - \frac{4}{x} \Rightarrow x = \frac{4}{20} = \frac{1}{5}$$

4 $a = -25x$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -25x$$

$$\frac{1}{2}v^2 = \int (-25x)\,dx = -\frac{25}{2}x^2 + A$$

At $x = 0$, $v = 40$

$$\frac{1}{2}\times 1600 = -0 + A \Rightarrow A = 800$$

$$\frac{1}{2}v^2 = -\frac{25}{2}x^2 + 800$$

$$v^2 = 1600 - 25x^2$$

When $v = 0$

$$25x^2 = 1600 \Rightarrow x^2 = 64 \Rightarrow x = \pm 8$$

So $AB = 16$ m

5 **a** $a = -kx^2$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -kx^2$$

$$\frac{1}{2}v^2 = \int (-kx^2)\,dx = -\frac{kx^3}{3} + A$$

At $x = 0,\ v = 16$

$$\frac{1}{2}\times 256 = -0 + A \Rightarrow A = 128$$

$$\frac{1}{2}v^2 = -\frac{kx^3}{3} + 128$$

When $v = 0,\ x = 20$

$$0 = -\frac{8000k}{3} + 128 \Rightarrow k = \frac{3\times 128}{8000} = \frac{3\times 16}{1000} = \frac{6}{125}$$

b From part **a**, $\frac{1}{2}v^2 = -\frac{6}{125}\times\frac{x^2}{3} + 128 \Rightarrow v^2 = 256 - \frac{4}{125}x^3$

At $x = 10$

$$v^2 = 256 - \frac{4}{125}\times 1000 = 256 - 32 = 224$$

$$v = \pm\sqrt{224} = \pm 4\sqrt{14}$$

The velocity of P at $x = 10$ is $\pm 4\sqrt{14}$ m s^{-1} as the particle will pass through this position in both directions.

6 $a = -8x^3$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = -8x^3$$

$$\frac{1}{2}v^2 = \int(-8x^3)\,\mathrm{d}x = -2x^4 + A$$

At $x = 2$, $v = 32$

$$\frac{1}{2}\times 1024 = A - 32 \Rightarrow A = 544$$

$$\frac{1}{2}v^2 = 544 - 2x^4$$

$$v^2 = 1088 - 4x^4$$

When $v = 8$

$$64 = 1088 - 4x^4 \Rightarrow x^4 = 256$$

$$x = 256^{\frac{1}{4}} = 4$$

7 **a** $a = 6\sin\frac{x}{3}$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = 6\sin\frac{x}{3}$$

$$\frac{1}{2}v^2 = \int\left(6\sin\frac{x}{3}\right)\mathrm{d}x = -18\cos\frac{x}{3} + A$$

At $x = 0$, $v = 4$

$$\frac{1}{2}\times 16 = -18 + A \Rightarrow A = 26$$

$$\frac{1}{2}v^2 = -18\cos\frac{x}{3} + 26$$

$$v^2 = 52 - 36\cos\frac{x}{3}$$

b The greatest value of v^2 occurs when $\cos\frac{x}{3} = -1$

The greatest value of v^2 is given by $v^2 = 52 + 36 = 88 \Rightarrow v = \pm\sqrt{88} = \pm 2\sqrt{22}$

So the greatest possible speed of P is $2\sqrt{22}\ \mathrm{m\,s^{-1}}$ ($\approx 9.38\ \mathrm{m\,s^{-1}}$)

8 $a = 2 + 3e^{-x}$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2 + 3e^{-x}$

$\frac{1}{2}v^2 = \int(2 + 3e^{-x})\,dx = 2x - 3e^{-x} + A$

At $x = 0$, $v = 2$

$\frac{1}{2} \times 4 = 0 - 3 + A \Rightarrow A = 5$

$\frac{1}{2}v^2 = 2x - 3e^{-x} + 5$

$v^2 = 4x - 6e^{-x} + 10$

At $x = 3$

$v^2 = 12 - 6e^{-3} + 10 = 21.701$ (3 d.p.)

$v = \sqrt{21.701\ldots} = 4.658$ (3 d.p.)

The velocity of P at $x = 3$ is 4.66 m s^{-1} (3 s.f.), in the direction of x increasing.

9 **a** $a = -\frac{4}{2x+1}$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{4}{2x+1}$

$\frac{1}{2}v^2 = \int\left(-\frac{4}{2x+1}\right)dx = -2\ln(2x+1) + A$

At $x = 0$, $v = 4$

$\frac{1}{2} \times 16 = -0 + A \Rightarrow A = 8$

$\frac{1}{2}v^2 = -2\ln(2x+1) + 8$

$v^2 = 16 - 4\ln(2x+1)$

At $x = 10$

$v^2 = 16 - 4\ln 21 = 3.8219$ (4 d.p.)

$v = 1.95\,\text{m s}^{-1}$ (3 s.f.)

The speed of P at $x = 10$ is $1.95\,\text{m s}^{-1}$ (3 s.f.)

b When $v = 0$

$0 = 16 - 4\ln(2x+1) \Rightarrow \ln(2x+1) = 4$

So $2x + 1 = e^4 \Rightarrow x = \frac{e^4 - 1}{2} = 26.8$ (3 s.f.)

10 a $a = x - \frac{4}{x^3}$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = x - \frac{4}{x^3}$$

$$\frac{1}{2}v^2 = \int (x - 4x^{-3})\,dx = \frac{x^2}{2} + 2x^{-2} + A = \frac{x^2}{2} + \frac{2}{x^2} + A$$

At $x = 1$, $v = 3$

$$\frac{1}{2}\times 9 = \frac{1}{2} + 2 + A \Rightarrow A = 2$$

$$\frac{1}{2}v^2 = \frac{x^2}{2} + \frac{2}{x^2} + 2$$

$$v^2 = x^2 + 4 + \frac{4}{x^2} = \left(x + \frac{2}{x}\right)^2$$

$$v = x + \frac{2}{x}$$

b The minimum value of v occurs when $\frac{dv}{dt} = a = 0$

$x - \frac{4}{x^3} = 0 \Rightarrow x^4 = 4 \Rightarrow x = \sqrt{2}$ (as P moves on the positive x-axis, $x > 0$)

At $x = \sqrt{2}$

$$v = \sqrt{2} + \frac{2}{\sqrt{2}} = \sqrt{2} + \sqrt{2} = 2\sqrt{2}$$

The least speed of P during its motion is $2\sqrt{2}\,\text{m s}^{-1}$

11 $a = -\left(10 + \frac{1}{4}x\right)$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -10 - \frac{1}{4}x$$

$$\frac{1}{2}v^2 = \int\left(-10 - \frac{1}{4}x\right)dx = -10x - \frac{x^2}{8} + A$$

At $x = 0$, $v = 15$

$$\frac{1}{2}\times 225 = -0 - 0 + A \Rightarrow A = \frac{225}{2}$$

$$\frac{1}{2}v^2 = -10x - \frac{x^2}{8} + \frac{225}{2}$$

$$v^2 = 225 - 20x - \frac{x^2}{4} = -\frac{x^2 + 80x - 900}{4} = -\frac{(x+90)(x-10)}{4}$$

$$v = 0 \Rightarrow x = 10,\ -90$$

As P is initially moving in the direction of x increasing, it reaches $x = 10$ before $x = -90$. The distance P moves before first coming to instantaneous rest is 10 m.

12 a $a = 6x^{\frac{1}{3}}$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = 6x^{\frac{1}{3}}$$

$$\frac{1}{2}v^2 = \int 6x^{\frac{1}{3}}\,\mathrm{d}x = \frac{6x^{\frac{4}{3}}}{\frac{4}{3}} + A = \frac{9}{2}x^{\frac{4}{3}} + A$$

$v^2 = 9x^{\frac{4}{3}} + B$, where $B = 2A$

At $x = 8$, $v = 12$

$$144 = 9 \times 16 + B \Rightarrow B = 0$$

$$v^2 = 9x^{\frac{4}{3}}$$

$$v = 3x^{\frac{2}{3}}$$

b $v = \frac{\mathrm{d}x}{\mathrm{d}t} = 3x^{\frac{2}{3}}$

Separating the variables and integrating

$$\int x^{-\frac{2}{3}}\,\mathrm{d}x = \int 3\,\mathrm{d}t$$

$$3x^{\frac{1}{3}} = 3t + C$$

When $t = 0$, $x = 8$

$$3 \times 2 = 0 + C \Rightarrow C = 6$$

$$3x^{\frac{1}{3}} = 3t + 6$$

$$x^{\frac{1}{3}} = t + 2$$

$$x = (t+2)^3$$

Challenge

$$a = \frac{1}{10}(25 - x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = \frac{25}{10} - \frac{x}{10}$$

$$\frac{1}{2}v^2 = \int\left(\frac{25}{10} - \frac{x}{10}\right)\mathrm{d}x = \frac{25x}{10} - \frac{x^2}{20} + A$$

The maximum value of v occurs when $\frac{\mathrm{d}v}{\mathrm{d}t} = a = 0$, $a = \frac{1}{10}(25 - x) = 0 \Rightarrow x = 25$

So at $x = 25$, $v = 12$

$$\frac{1}{2}12^2 = \frac{25 \times 25}{10} - \frac{25^2}{20} + A \Rightarrow A = 72 - \frac{625}{20} = \frac{288 - 125}{4} = \frac{163}{4}$$

Hence $\frac{1}{2}v^2 = \frac{25x}{10} - \frac{x^2}{20} + \frac{163}{4}$, so $v^2 = \frac{25x}{5} - \frac{x^2}{10} + \frac{163}{2} = \frac{1}{5}\left(25x - \frac{x^2}{2}\right) + \frac{163}{2}$